THE DEVELOPMENT OF A FRAMEWORK OF GROWTH POINTS TO MONITOR STUDENTS' COMPREHENSION OF ALGEBRA IN GRADES 7-9

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This paper is proposed to stimulate discussion in the area of algebra. It reviews relevant literature in algebra and proposes a framework of growth points to inform future research and to enable monitoring algebraic development through the early years of secondary school through understanding rather than through outcomes. It is work in progress in preliminary stages with some areas still to complete.

The teaching and learning of Algebra and its place in the curriculum has been of interest and debate to curriculum writers for many years. Grimison (1995) describes its introduction to the syllabi of the British Empire in the late nineteenth century and subsequent lack of change through a large part of the twentieth century. Whatever the reasons for its introduction, and the rights and wrongs of it as appropriate curriculum for all, it is part of the curriculum in Australian schools and indeed in the rest of the world.

However, the conflicts about the teaching of algebra have continued. There is general agreement that there are cognitive difficulties in moving from arithmetic to algebra and the teaching and learning of algebra is a major area of concern in mathematics education. Algebra still occupies a large part of the syllabi in secondary schools.

Some changes were introduced into our syllabi in the nineteen sixties and more recently, following *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991), most States have developed new curriculum documents (e.g., Board of Studies, 1995). The National Statement provided a framework that divides algebra into the sub-strands expressing generality, functions and equations. The framework is an outcomes based structure that manages to give general outcomes showing broad development.

While this framework provides a useful scaffolding for curriculum development and reflects ideas of development, it does not effectively provide a means of monitoring children's understanding of algebra in grades 7-9, based on the considerable work of researchers in the area of cognition in algebra. What is needed is a framework built on the development of students' thinking rather than outcomes.

Researchers have understandably focussed on particular aspects of algebra such as the modelling used when solving word problems (e.g. Lemut & Greco, 1998; MacGregor & Stacey, 1993, 1998), the understanding of the equals sign (e.g. Kieran, 1981; Pillay, Wilss & Boulton-Lewis, 1998), the translation from tabular form to symbolic form (e.g. Redden, 1994; Ryan & Williams, 1998; Warren, 1998), the solution of linear equations (e.g. Linchevski & Herscovics, 1994; MacGregor & Stacey, 1995) and functions and graphs (e.g. Herscovics, 1989; Swan, 1988).

Algebra is not a simple domain. There are many different aspects associated with algebraic cognition. In algebra we operate with a range of mathematical sign systems such as graphs, algebra code and tables on a variety of mathematical objects (Filloy & Sutherland, 1996). Kaput (1989) suggests four sources of meaning in relation to mathematical sign systems:

- translation within a system such as symbolic manipulation to alter an equation;
- translation across systems such as from a graph to a table;
- translation between mathematical sign systems and non mathematical sign systems such as natural language;

- the process of abstraction which occurs as students evolve mathematical sign systems through consolidation, simplification, generalisation and reification of the actions, routines and representations which produced intermediate sign systems in the teaching sequences.

As the mathematical sign systems develop and the abstractions occur there is an interaction between many of the aspects of development. For example the understanding of the equals sign develops alongside understanding of aspects of linear equations in an interactive way rather than linearly. Within each area a hierarchical development can be seen. While Kaput's sources of meaning are useful to indicate the scope of development they are not directly hierarchical as developments occur in each of the first three translation areas concurrently. What is critical is that the development is moving always towards abstraction and recognition of the underlying structure. In order to monitor students' progress through algebra in the early years in particular, a framework is needed which highlights some of the growth points in development and shows the hierarchical structure while at the same time acknowledges the separate developments in different areas.

Many researchers have been concerned with the move from arithmetic to algebra and in particular the cognitive gap that exists between the two (e.g. Bednarz, Radford, Janvier & Lepage, 1992). Algebra is not simply an extension of the numerical domain and a question of symbolism. Rather it is a way to manipulate relations and based on analysis. Algebra uses the same words and symbols as arithmetic but the meaning is not always the same. The objects are essentially different. Consider 3+5 and d+5. The first can be written more simply as 8, and in the process the actual parts can no longer be seen. d+5 remains as is, with the two original parts still open to view. Many students try to combine these to $d5$ or $5d$ or even 9 in an attempt at closure. Acceptance of lack of closure (Collis, 1975) is an important growth point in algebraic development. Understanding this is important in the development of the necessary abstractions.

Table 1
Generalised Arithmetic Laws

	Generalised arithmetic laws	Research Support
1	Basic arithmetical understanding of operations: shown by the application of the four basic operations to binary problems.A high degree of facility with decimals and fractions is not necessary, students should be able to operate with simple decimals and fractions	Some support: Cooper et al 1997, Boulton-Lewis et al, 1997, Pillay et al, 1998, Horne, 1994; Kieran,1989; Linchevski & Herscovics, 1994
2	Commutativity: shown in the understanding of the differences between 2+9 and 9+2, and 6÷3 and 3÷6. Many students can successfully perform simple operations but have not yet realised the differences in structure	
3	Order of operations: This extends the operations from level 1 to include more than one operation	
4	Distributive law: This requires more of a structural understanding than simply an arithmetic understanding and should include division and subtraction as well as addition and multiplication	

Table 2
Concept of "="

	Concept of "="	Research Support
1	Equals meaning "now do something" or "find the answer". For example students who write 10 + 3 = 13 – 2 = 11 are thinking of the equals as an indicator for action or a punctuation mark	Some support: Kieran, 1981; Pillay et al, 1998.
2	Equals means one expression equals the other if you work them out (but in terms of numerical calculations) (Willingness to have expressions on both sides of the = but still seeing it as "if you work this out it equals that".)	
3	Equivalence of expressions (including the transitive and symmetric nature)	

The understanding of the equals sign and the understanding of operations and the laws of arithmetic (and algebra) are two aspects critical to further development in algebra. Often success in algebra is hampered by poor arithmetical skills and understanding (Horne, 1994; Kieran, 1989; Linchevski & Herscovics, 1994). The development of operations and rules associated with arithmetic, and the development of understanding of the equals sign from a sign that an action is needed to an understanding of equivalence with symmetric and transitive properties, occur concurrently with interaction between these separate but related areas and other areas such as the solution of equations. In the proposed framework, these areas have been indicated by *Generalised arithmetic laws* in Table 1 and *Concept of "="* in Table 2.

Table 3
Identity

	Identity	Research Support
1	Understanding $3x$ and $x + 2$ with x representing a particular unknown number. For example $3x = 6$ if $x = 2$	Support for some parts: Cooper et al, 1997; Collis, 1995.
2	Willingness to accept lack of closure	
3	Combination of like terms with idea that x could be any unknown number	
4	Ability to find equivalent expressions using a range of operations where x could be many numbers	
5	Ability to find equivalent expressions with the recognition that x is an object which may be representing other objects such as a number or an expression like $(a + 2)$	
*	Use of strategies to check equivalence	

* It is not clear how checking strategies fit in hierarchy

The understanding of the x in algebra has had an associated hierarchy of understanding initially proposed by Küchemann (1981). It is not necessary in development though for students to move through all of the inadequate concepts in this structure. The x of algebra can be considered an as yet unknown, a general number, or a variable – which can be symbolized and operated on as an object (as if it were a number) (Wheeler, 1996). These concepts of the x link to equations, identities, properties, and functions and relations respectively (Janvier, 1996) and are all necessary to a full understanding of algebra. These understandings develop in all three of Kaput's translations within a sign system, between sign systems and with natural language. For this reason rather than consider the development of understanding of the "x" as one area, it occurs across a number of areas. Operating within an algebraic symbolic sign system involves manipulation of the symbols. This includes combination of symbols, understanding of 3x, the idea of acceptance of lack of closure (Collis, 1975) and the understanding of equivalence of expressions. This area is shown in Table 3.

Another area operating within the symbolic sign system is the solution of equations where students manipulate the algebraic equation in order to find a solution. Classified here as Equations, and shown in Table 4, many researchers have been interested in this area which is closely linked also to the conception of the equals sign. A few researchers have developed hierarchies for linear equations that indicate the development and the cognitive complexities of the situations. Pillay, Wilss and Boulton-Lewis (1998) have developed a schema from arithmetic through pre-algebra to algebra and have based part of this on the cognitive load faced by the students. During the transition from arithmetic to algebra they suggest that the equals sign changes from meaning each side has the same value to equivalence. They also at the same time see equations moving from purely numeric to simple linear one variable equations to linear equations where there is more than one unknown or variable. Their structure recognises the increasing complexities of the tasks.

Table 4
Equations

	Equations	Research Support
1	Numerical understanding 1 step or two steps using box (Completion of fill in gaps in arithmetic (or use of box) in one step equation format.)	Some support: Linchevski & Herscovics, 1994; MacGregor & Stacey, 1995; Pillay et al, 1998
2	Solution of one step and 2 step equations using algebraic symbols with whole numbers by trial/knowledge of number facts	
3	Use of backtracking to solve 2 or more step equations with whole numbers and inverse ideas	
4	Solution of equations with non whole number solutions	
5	Use of (inverse operations/) balance to solve 2 step equations	
6	Simple equations with more than one occurrence of the variable on one side.	
7	Simple equations with the variable appearing on both sides	

One difficulty is that within a particular area growth points can be chosen to show an order. There will be other aspects of learning as well but the idea of these growth points is to choose key aspects of development. The growth in any area is from limited arithmetic and concrete understanding towards abstract and structural understanding. It is not clear for students when each of these areas begins and ends, nor is it always clear which growth points in one area should or do precede a growth point in another area.

There are other areas. When students translate between systems the ideas of functional development and variable are raised. Students' early development, particularly as it is now presented in many syllabi, includes pattern recognition in the translation from table form of a function or relation to symbolic form. There has been much research in this area. In particular in Australia, Redden (1994), Warren (1998) and Ryan & Williams (1998) have all independently investigated student's development of rules from tables. Early development seems to involve students seeing the pattern in terms of the previous term in the sequence rather than being dependent on the independent variable x. Ryan and Williams (1998) suggested that the tendency to introduce the table in an ordered fashion with x values being sequential tends to focus the attention on this aspect rather than the relationship between the variables. While there are minor differences in the research findings in this field, generally there is agreement that early stages include the addition of a constant and the consideration of the rule as an iterative rule with the x value being the number of the term. In the later stage students can develop an algebraic rule, and move between the two variables in the relationship comfortably. Garcia-Cruz and Martinón (1998) have developed a three level hierarchy for this type of problem based on the degree of action and generalisation from procedural activity through procedural understanding and local generalisation to conceptual understanding and global generalisation. While this structure indicates important directions of development, it has used fewer growth points than are indicated by the body of research available. This area has been classified as *Function – table* in Table 5.

Table 5
Function – Table

	Function - Table	Research Support
1	Finding next term in a table with sequential independent variable values but unable to give explanation	Some support: Redden, (1994); Ryan & Williams (1998); Warren (1998); Garcia-Cruz and Martinón (1998)
2	Explaining next term in terms of previous terms but not recognising the independent variable	
3	Ability to complete tables (linear expressions) with missing terms by working from previous terms	
4	Finding terms in a table not presented sequentially but based on dependent variable	
5	Ability to express a linear rule symbolically	
*	Use of non integer numbers in table	

* It is not clear where in this understanding students start to be able to use a range of number types.

Table 6.
Function – graphs

	Function – graphs	Research Support
1	Reading a value from a graph with labelled axes	Support for some parts: Swan, 1988; Herscovics, 1989 The order is not clear from the literature.
2	Reading a point from a graph and plotting points on a cartesian grid	
3	General interpretation of a graph quantity as getting less or getting more	
4	Recognition of table to graph and vice versa	
5	Plotting a linear rule expressed algebraically	
6	Recognition of the link between gradient (steepness) and the rule	
7	Recognition of the link between c and the cutting of the y-axis	
8	Recognition of vertical and horizontal lines	

Another between systems aspect involves the translation involving graphs, classified here as *Function - graphs* in Table 6. Much research on functions and graphs has been at a higher level and, in recent years, has involved the impact of the use of technology. Swan (1988) indicated students' lack of understanding of graphs both at the basic level of interpretation of points and at the more dynamic level of considering functions and graphs as showing variation. Nemirovsy (1996), in responding to some work on using a technological environment to teach functions, discusses the issue of a point-wise approach versus a variation approach. Children using motion detectors to develop function ideas show a relational understanding although researchers such as Kieran, Boileau and Garançon (1996) have argued for a discrete mathematical approach to function as a set of points rather than a continuous variation. While there are extra levels of complexity in the full understanding of graphs as continuous, containing an infinite number of points, as Nemirovsky notes, children show a relational understanding before their understanding of points is fully developed so one does not necessarily precede the other. More work is needed on cognition in this area.

In Australian Education Council (1991), equations and inequalities are considered in the one sub-strand as closely related. There is little research to indicate student's cognitive development in this area of inequalities, but perhaps it also should be included in a framework. This is indicated in Table 7.

Table 7
Other areas to be included

	Inequalities	Research support
1*	Recognition of greater than and less than symbols and correct use in arithmetic expressions.	
2*	Understands that there are many correct answers to an algebraically expressed inequality with one operation.	
*	Further stages are needed	
	Modelling	**Research support**
	This section is still needed and there have been many researchers working with modelling and problem solving and how students use algebraic formulations of the problem. This area is concerned with the translation from natural language and situation to algebraic representation.	Some of the many researchers include: MacGregor & Stacey, 1993, 1998; Lemut & Greco, 1998; Brito Lima and Da Rocha Falcão (1997)

* This is purely speculative

Problem solving and mathematical modelling are a large part of algebra and for many provide it with its *raison d'être*. This involves the translation from natural language into algebraic symbolisation, although many problems that are intended to require the use of algebra can be solved with no reference to algebraic approaches. There are many nuances in natural language and many different ways of presenting the same problem. MacGregor and Stacey (1998) found that one form of a question on unequal partitioning directed the solution to an arithmetic solution through one cognitive model, while another form of the same question led more easily to an algebraic formulation of the problem. There was actually no significant difference between the two forms of the questions in the success experienced by the students, or in the degree of algebraic formulation finally reached but rather a tendency for the form of the question to direct the cognitive model initially in a particular direction. Brito Lima and Da Rocha Falcão (1997), using different basic problem forms, also found that the form of the question did not lead to significantly different results or ways of approaching the problem. In earlier work MacGregor and Stacey (1993) reported on a study which looked at students' syntactical translation of language into symbolic form in the study of equations. Contrary to their expectations they found that students did not frequently use syntactical translation in the development of symbolic representation of equations. This translation from natural language in problem solving situations to algebraic form is a crucial part of algebraic understanding. A hierarchy in this area would be expected to move from simple syntactical translations to more complex modelling skills. The growth points here may be classified in terms of levels of complexity. This is also indicated in Table 7.

All of these proposed hierarchies could be expanded to include many more details and fine differences. However, the purpose of this is to provide a framework which focuses on understandings as students move from limited concepts and arithmetic and concrete understandings to broader concepts and abstractions. For this reason there has been an attempt to distill the key ideas in a framework that broadly covers the area but is simple to use.

CONCLUSION

The proposed structure is intended to focus on student's understanding rather than on their skill at reproducing taught methods. It should provide a means of monitoring students' development in moving towards the abstraction of algebra. It has also highlighted areas where there does not appear to be a clear understanding of the cognitive development.

Such a framework needs a considerable amount of research but could provide a useful structure for other researchers and for teachers in schools who are trying to assist students to learn algebra.

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